



Low-energy moments of non-diagonal quark current correlators at four loops

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Abstract

We compute the leading four physical terms in the low-energy expansions of heavy–light quark current correlators at four-loop order. As a by-product we reproduce the corresponding top-induced non-singlet correction to the electroweak ρ parameter.

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1. Introduction

Two-point correlation functions of heavy–light quark currents have found use in a number of phenomenological applications. One example is the prediction of corrections to the electroweak ρ parameter [1–3], where the flavour non-diagonal correlator of vector currents is required for vanishing external momentum. Another important class of applications is the sum-rule determination of meson decay constants (see e.g. [4,5]). Here, the absorptive part of the respective correlators above the production threshold is needed.

Progress in lattice simulation may allow precise determinations of even more QCD parameters. For instance, the values of the strong coupling constant, the charm quark mass and the bottom quark mass have been determined with high accuracy from moments of heavy–heavy

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correlators in [6,7]. In these analyses, moments of flavour diagonal currents have been determined on the lattice choosing a frame where the spatial momentum of the correlators vanishes. The values of the quark masses and the coupling constant are then extracted by equating these moments to their counterparts calculated in perturbation theory at the four-loop order [8–14].

The methodology is thus similar to traditional quarkonium sum rules [15–17], but using lattice moments in place of moments of the experimentally measured hadronic R ratio. While for the sum rules only the correlator of vector currents can be used, there is no such restriction for the lattice simulation. In fact, in [6] different Lorentz structures were considered, with the most precise results stemming from pseudo-scalar currents. Furthermore, also correlators of heavy–light currents could be used to extract the values of the charm and bottom quark masses and possibly the strong coupling constant [18]. To be competitive with the analyses for the heavy–heavy case the corrections to the perturbative moments of the heavy–light current correlators have to be known up to four loops. These corrections are presented in this work.

Given their usefulness, perturbative corrections to heavy–light correlators have been studied quite intensively and analytic results up to two loops have been known for many years [19,20]. While the three-loop correction is not known analytically, many terms in expansions in both the low-energy and the high-energy limit have been calculated in [21–23]. Combining these with the behaviour near threshold, accurate approximations for arbitrary kinematics have been constructed [21,22]. In the low-energy region also corrections due to a non-vanishing light quark mass are known [24,25].

The four-loop corrections remain mostly unknown. In the high-energy region the leading term is equal to the non-singlet part of the corresponding diagonal correlator, which has been computed for both scalar and vector currents [26–28]. In the low-energy region, conversely, there is no such simple correspondence between diagonal and non-diagonal correlators. The vector correlator in the limit of vanishing external momentum constitutes a central ingredient in the determination of non-singlet four-loop corrections to the ρ parameter, which have been calculated in [2,3].

In this work we present the four-loop corrections to the low-energy expansions of both scalar and vector heavy–light quark current correlators up to the eighth power of the external momentum. After introducing our conventions in Section 2, we briefly describe the calculational setup and present our results in Section 3. Section 4 describes the re-calculation of the top-induced contributions to the electroweak ρ parameter, which constitutes an important consistency check. We conclude in Section 5.

2. Conventions

The correlators of heavy–light vector and scalar currents are defined as

$$\Pi_{\mu\nu}(q) = i \int dx e^{iqx} \langle 0 | j_\mu(x) j_\nu(0) | 0 \rangle, \quad (1)$$

$$\Pi(q) = i \int dx e^{iqx} \langle 0 | j(x) j(0) | 0 \rangle \quad (2)$$

with the vector current $j_\mu(x) = \bar{\psi}(x) \gamma_\mu \chi(0)$ and the scalar current $j(x) = \bar{\psi}(x) \chi(0)$. We consider a heavy quark ψ with the pole mass m and a massless light quark χ . It should be noted that in the limit of a vanishing light-quark mass the correlators of two axial-vector or pseudo-scalar currents coincide with the vector and scalar correlators, respectively.

It is convenient to introduce polarisation functions

$$\Pi_{\mu\nu}(q) = (-q^2 g_{\mu\nu} + q_\mu q_\nu) \Pi^v(q^2) + q_\mu q_\nu \Pi_L^v(q^2), \quad (3)$$

$$\Pi(q) = q^2 \Pi^s(q^2). \quad (4)$$

In the following we will not consider the longitudinal polarisation $\Pi_L^v(q^2)$. The perturbative expansions of $\Pi^\delta(q^2)$ with $\delta = v, s$ up to four loops read

$$\Pi^\delta(q^2) = \Pi^{\delta,(0)}(q^2) + \frac{\alpha_s}{\pi} C_F \Pi^{\delta,(1)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^2 \Pi^{\delta,(2)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^3 \Pi^{\delta,(3)}(q^2) + \dots \quad (5)$$

Being interested in the limit $q^2 \rightarrow 0$, we can expand the coefficients in the above series as

$$\Pi^{\delta,(i)}(q^2) = \frac{3}{16\pi^2} \sum_{n=-1}^{\infty} C_n^{\delta,(i)} z^n = \frac{3}{16\pi^2} \sum_{n=-1}^{\infty} \bar{C}_n^{\delta,(i)} \bar{z}^n, \quad (6)$$

where we have used the abbreviations $z = q^2/m^2$, $\bar{z} = q^2/\bar{m}^2$ with \bar{m} denoting the mass of the heavy quark in the $\overline{\text{MS}}$ scheme. Note that the coefficients with $n = -1, 0$ still contain poles in the limit $\epsilon = (4-d)/2 \rightarrow 0$. In physical observables these have to be cancelled by the wave-function and mass renormalisations of the particles (e.g. W bosons) coupling to the respective current. In the following we will describe the calculation of the coefficients $C_n^{\delta,(3)}$, $\bar{C}_n^{\delta,(3)}$ for $n \leq 4$.

3. Calculation and results

First, the four-loop diagrams contributing to the polarisation functions are generated with QGRAF [29]. In the following steps we perform several algebraic manipulations with the help of TFORM [30,31]. As a first simplification, we apply partial fractioning to denominators that differ only by their mass and the external momentum q , i.e. we use

$$\frac{1}{p^2} \frac{1}{(p \pm q)^2 - m^2} = \frac{1}{q^2 \pm 2pq - m^2} \left(\frac{1}{p^2} - \frac{1}{(p \pm q)^2 - m^2} \right). \quad (7)$$

Since we will perform an expansion in q the prefactor on the right-hand side has no influence on the tadpole topology of the considered diagram. Performing partial fractioning before the identification of the diagram topologies greatly reduces both the number and the complexity of the topologies that have to be considered. Using the algorithm described in Appendix A we map the resulting diagrams onto 28 topologies.

Next, colour factors are calculated using the FORM [30] package color [32]. We choose a routing for the external momentum q which minimizes the number of propagators depending on q . After this we evaluate the traces over gamma matrices and perform a Taylor expansion in q . The scalar integrals we obtain after tensor reduction and the elimination of reducible scalar products are reduced to master integrals using a private implementation¹ [35] of Laporta's algorithm [36]. All required master integrals are known analytically or numerically [37–39].

For the presentation of our results we impose the overall renormalisation condition $\Pi^v(0) = \Pi^s(0) = 0$. The corresponding divergent subtraction terms are listed in Appendix B. For the remaining coefficients according to equation (6) we obtain²

¹ The implementation is written in C++ and uses GiNaC [33] and fermat [34].

² The moments are presented in a form that yields five significant digits for QCD with $n_f \leq 5$ light flavours. All results are also available at high precision as supplementary material to the online publication of this work.

$$\begin{aligned}
C_1^{v,(3)} = & + 14.5508 C_A^2 C_F + 8.4892 C_A C_F^2 + 0.351 C_F^3 \\
& - 0.2294 C_A C_F T_F n_h - 0.6242 C_F^2 T_F n_h \\
& - 12.56835 C_A C_F T_F n_l - 3.07525 C_F^2 T_F n_l \\
& + 0.107 C_F T_F^2 n_h^2 + 0.14 C_F T_F^2 n_h n_l + 1.91917 C_F T_F^2 n_l^2, \tag{8}
\end{aligned}$$

$$\begin{aligned}
C_2^{v,(3)} = & + 7.39116 C_A^2 C_F + 5.65943 C_A C_F^2 + 0.80504 C_F^3 \\
& + 0.0683 C_A C_F T_F n_h - 0.3114 C_F^2 T_F n_h \\
& - 6.0806 C_A C_F T_F n_l - 2.2303 C_F^2 T_F n_l \\
& + 0.008 C_F T_F^2 n_h^2 - 0.0052 C_F T_F^2 n_h n_l + 0.9442 C_F T_F^2 n_l^2, \tag{9}
\end{aligned}$$

$$\begin{aligned}
C_3^{v,(3)} = & + 4.42563 C_A^2 C_F + 3.86666 C_A C_F^2 + 0.73105 C_F^3 \\
& + 0.0448 C_A C_F T_F n_h - 0.1713 C_F^2 T_F n_h \\
& - 3.57037 C_A C_F T_F n_l - 1.5461 C_F^2 T_F n_l \\
& + 0.0017 C_F T_F^2 n_h^2 - 0.005 C_F T_F^2 n_h n_l + 0.56396 C_F T_F^2 n_l^2, \tag{10}
\end{aligned}$$

$$\begin{aligned}
C_4^{v,(3)} = & + 2.90512 C_A^2 C_F + 2.7515 C_A C_F^2 + 0.5965 C_F^3 \\
& + 0.0278 C_A C_F T_F n_h - 0.104 C_F^2 T_F n_h \\
& - 2.31867 C_A C_F T_F n_l - 1.10502 C_F^2 T_F n_l \\
& + 0.0006 C_F T_F^2 n_h^2 - 0.003 C_F T_F^2 n_h n_l + 0.3708 C_F T_F^2 n_l^2, \tag{11}
\end{aligned}$$

$$\begin{aligned}
C_1^{s,(3)} = & + 1.6424 C_A^2 C_F + 1.65318 C_A C_F^2 + 1.41042 C_F^3 \\
& - 1.39916 C_A C_F T_F n_h + 0.5551 C_F^2 T_F n_h \\
& - 3.129699 C_A C_F T_F n_l + 0.556834 C_F^2 T_F n_l \\
& + 0.376 C_F T_F^2 n_h^2 + 0.65308 C_F T_F^2 n_h n_l + 0.441495 C_F T_F^2 n_l^2, \tag{12}
\end{aligned}$$

$$\begin{aligned}
C_2^{s,(3)} = & + 5.66925 C_A^2 C_F + 5.36995 C_A C_F^2 + 2.1099 C_F^3 \\
& - 0.0476 C_A C_F T_F n_h + 0.1338 C_F^2 T_F n_h \\
& - 5.00716 C_A C_F T_F n_l - 1.60465 C_F^2 T_F n_l \\
& + 0.037 C_F T_F^2 n_h^2 + 0.0314 C_F T_F^2 n_h n_l + 0.711301 C_F T_F^2 n_l^2, \tag{13}
\end{aligned}$$

$$\begin{aligned}
C_3^{s,(3)} = & + 4.18695 C_A^2 C_F + 4.9201 C_A C_F^2 + 2.0783 C_F^3 \\
& + 0.0196 C_A C_F T_F n_h - 0.0103 C_F^2 T_F n_h \\
& - 3.5077 C_A C_F T_F n_l - 1.7089 C_F^2 T_F n_l \\
& + 0.009 C_F T_F^2 n_h^2 + 0.0006 C_F T_F^2 n_h n_l + 0.5215 C_F T_F^2 n_l^2, \tag{14}
\end{aligned}$$

$$\begin{aligned}
C_4^{s,(3)} = & + 2.945114 C_A^2 C_F + 3.81782 C_A C_F^2 + 1.6905 C_F^3 \\
& + 0.01994 C_A C_F T_F n_h - 0.0347 C_F^2 T_F n_h \\
& - 2.41729 C_A C_F T_F n_l - 1.3927 C_F^2 T_F n_l \\
& + 0.0034 C_F T_F^2 n_h^2 - 0.0016 C_F T_F^2 n_h n_l + 0.36976 C_F T_F^2 n_l^2, \tag{15}
\end{aligned}$$

where we have set the renormalisation scale μ to the on-shell mass m . We follow the usual convention for the colour factors with $C_A = 3$, $C_F = 4/3$, $T_f = 1/2$ for QCD. The number of light (massless) quark flavours is denoted by n_l , whereas n_h stands for the number of heavy flavours.

If we choose to express the polarisation functions in terms of the $\overline{\text{MS}}$ mass \bar{m} at the scale $\mu = \bar{m}$ and $\alpha_s(\bar{m})$, we arrive at

$$\begin{aligned}\bar{C}_1^{v,(3)} = & -1.2994791 C_A^2 C_F + 1.20957 C_A C_F^2 + 0.537098 C_F^3 \\ & -1.75125 C_A C_F T_F n_h + 1.29201 C_F^2 T_F n_h \\ & + 0.530618 C_A C_F T_F n_l - 0.0193 C_F^2 T_F n_l \\ & - 0.0853 C_F T_F^2 n_h^2 + 0.07322 C_F T_F^2 n_h n_l - 0.0389 C_F T_F^2 n_l^2, \quad (16)\end{aligned}$$

$$\begin{aligned}\bar{C}_2^{v,(3)} = & -1.0623284 C_A^2 C_F + 1.035507 C_A C_F^2 + 0.1608 C_F^3 \\ & - 0.74336 C_A C_F T_F n_h + 0.72663 C_F^2 T_F n_h \\ & + 0.905515 C_A C_F T_F n_l - 0.46186 C_F^2 T_F n_l \\ & - 0.0944796 C_F T_F^2 n_h^2 - 0.04078 C_F T_F^2 n_h n_l - 0.10011 C_F T_F^2 n_l^2, \quad (17)\end{aligned}$$

$$\begin{aligned}\bar{C}_3^{v,(3)} = & -0.8577978 C_A^2 C_F + 1.1607802 C_A C_F^2 - 0.1496539 C_F^3 \\ & - 0.4624948 C_A C_F T_F n_h + 0.520167 C_F^2 T_F n_h \\ & + 0.7959545 C_A C_F T_F n_l - 0.631599 C_F^2 T_F n_l \\ & - 0.06236 C_F T_F^2 n_h^2 - 0.027228 C_F T_F^2 n_h n_l - 0.08873276 C_F T_F^2 n_l^2, \quad (18)\end{aligned}$$

$$\begin{aligned}\bar{C}_4^{v,(3)} = & -0.717803 C_A^2 C_F + 1.286187 C_A C_F^2 - 0.40493 C_F^3 \\ & - 0.320038 C_A C_F T_F n_h + 0.41065 C_F^2 T_F n_h \\ & + 0.67538025 C_A C_F T_F n_l - 0.72124 C_F^2 T_F n_l \\ & - 0.04337 C_F T_F^2 n_h^2 - 0.018157 C_F T_F^2 n_h n_l - 0.076781 C_F T_F^2 n_l^2, \quad (19)\end{aligned}$$

$$\begin{aligned}\bar{C}_1^{s,(3)} = & +1.642401 C_A^2 C_F - 0.510074 C_A C_F^2 + 1.41042 C_F^3 \\ & - 1.39916 C_A C_F T_F n_h + 1.34177 C_F^2 T_F n_h \\ & - 3.1297 C_A C_F T_F n_l + 1.343472 C_F^2 T_F n_l \\ & + 0.3759 C_F T_F^2 n_h^2 + 0.65308 C_F T_F^2 n_h n_l + 0.4415 C_F T_F^2 n_l^2, \quad (20)\end{aligned}$$

$$\begin{aligned}\bar{C}_2^{s,(3)} = & +0.38582 C_A^2 C_F + 0.31725 C_A C_F^2 + 0.916757 C_F^3 \\ & - 0.55487 C_A C_F T_F n_h + 0.80004 C_F^2 T_F n_h \\ & - 0.640838 C_A C_F T_F n_l + 0.44938 C_F^2 T_F n_l \\ & - 0.02698 C_F T_F^2 n_h^2 + 0.0092 C_F T_F^2 n_h n_l + 0.05861 C_F T_F^2 n_l^2, \quad (21)\end{aligned}$$

$$\begin{aligned}\bar{C}_3^{s,(3)} = & -0.039796 C_A^2 C_F + 0.48671411 C_A C_F^2 + 0.4018122 C_F^3 \\ & - 0.38621 C_A C_F T_F n_h + 0.46308 C_F^2 T_F n_h \\ & - 0.014653 C_A C_F T_F n_l + 0.042405 C_F^2 T_F n_l \\ & - 0.0422 C_F T_F^2 n_h^2 - 0.01723 C_F T_F^2 n_h n_l - 0.00067 C_F T_F^2 n_l^2, \quad (22)\end{aligned}$$

$$\begin{aligned}
\bar{C}_4^{s,(3)} = & -0.224943 C_A^2 C_F + 0.565757 C_A C_F^2 + 0.067439 C_F^3 \\
& - 0.284427 C_A C_F T_F n_h + 0.3319473 C_F^2 T_F n_h \\
& + 0.2025025 C_A C_F T_F n_l - 0.162445 C_F^2 T_F n_l \\
& - 0.035056 C_F T_F^2 n_h^2 - 0.014915 C_F T_F^2 n_h n_l - 0.021858 C_F T_F^2 n_l^2.
\end{aligned} \quad (23)$$

4. The ρ parameter

To verify the correctness of our calculation we have performed a number of cross checks. Obviously, our results are UV-finite. We have also performed an expansion up to linear order in the gauge parameter and verified that the gauge dependence cancels in the coefficients $\bar{C}_1^{v,(3)}$, $\bar{C}_1^{s,(3)}$. The strongest check, however, is the comparison to the known four-loop non-singlet corrections to the ρ parameter [2,3].

The electroweak ρ parameter has been introduced in Ref. [40]. Considering only QCD corrections it can be written as

$$\rho = 1 + \delta\rho \quad (24)$$

with

$$\delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}, \quad (25)$$

where $\Pi_{ZZ}(0)$ and $\Pi_{WW}(0)$ denote the self energies of Z and W boson, respectively.

In order to calculate the contribution from the Z -boson self energy to the ρ parameter we also need the leading moment of the flavour diagonal correlator. To this end we introduce $\Pi_{\text{diag}}^a(q^2)$ similar to Eq. (1) but with the heavy-heavy axial current

$$\tilde{j}_a^\mu = \bar{\psi} \gamma_5 \gamma^\mu \psi, \quad (26)$$

and the moments

$$\Pi_{\text{diag}}^{a,(3)}(q^2) = \frac{3}{16\pi^2} \sum_{n=-1}^{\infty} C_n^{a,(3)} \left(\frac{q^2}{4m^2} \right)^n. \quad (27)$$

In what follows we will only consider the top-induced four-loop correction to ρ , corresponding to ρ_3 in the expansion

$$\delta\rho = 3x_t \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^i \rho_i, \quad x_t = \frac{\sqrt{2} G_F m_t^2}{16\pi^2}. \quad (28)$$

The corresponding corrections to the Z and W self energies then read

$$\begin{aligned}
\frac{\Pi_{ZZ}^{(3)}(0)}{M_Z^2} = & 3x_t \left[\left(1 - \frac{1}{d} \right) C_{-1,\text{diag}}^{a,(3)} - \frac{1}{d} C_{L,-1,\text{diag}}^{a,(3)} \right] \\
& + \text{singlet terms},
\end{aligned} \quad (29)$$

and

$$\frac{\Pi_{WW}^{(3)}(0)}{M_W^2} = 3x_t \left[\left(1 - \frac{1}{d} \right) C_{-1}^{v,(3)} - \frac{1}{d} C_{L,-1}^{v,(3)} \right], \quad (30)$$

where the higher-order corrections $\Pi_{ZZ}^{(3)}$, $\Pi_{WW}^{(3)}$ are defined in analogy to equation (5). $C_{L,-1}^{v,(3)}$ and $C_{L,-1,\text{diag}}^{a,(3)}$ denote the moments with $n = -1$ of the respective longitudinal polarisation functions; from an explicit calculation we obtain

$$C_{L,-1}^{v,(3)} = -C_{-1}^{v,(3)}, \quad C_{L,-1,\text{diag}}^{a,(3)} = -C_{-1,\text{diag}}^{a,(3)}. \quad (31)$$

Note that in the non-diagonal case the vector and axial-vector correlators coincide and that the (-1) -th moment of the diagonal vector correlator vanishes. The contributions from W - and Z -boson self energies are divergent on their own and only their sum is finite. The singlet terms calculated in Ref. [1] are finite on their own and we do not repeat them here. Using the results given in Appendix B we obtain in the $\overline{\text{MS}}$ scheme

$$\bar{\rho}_{3,\text{non-singlet}} = \bar{C}_{-1,\text{diag}}^{a,(3)} - \bar{C}_{-1}^{v,(3)} = 1.60667, \quad (32)$$

and after converting to the on-shell scheme

$$\rho_{3,\text{non-singlet}} = -101.083, \quad (33)$$

in full agreement with the results in the literature [2,3].

5. Conclusion

We have calculated the four-loop QCD corrections to the low-energy moments of flavour non-diagonal current correlators up to $n = 4$. Our results are valid for (axial-)vector and (pseudo-)scalar currents in the limit of a vanishing light-quark mass. As a by-product we have confirmed the results for the non-singlet correction to the electroweak ρ parameter first obtained in [2,3]. In combination with lattice simulations, our results can be used for the precision determination of heavy-quark masses. Furthermore, they can serve as an ingredient in the approximate reconstruction of the four-loop corrections for arbitrary external momenta. For the latter application, however, more input from other kinematic regions is still required.

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Appendix A. Symmetrisation

The closely related problems of symmetrisation and mapping diagrams to topologies are ubiquitous in multiloop calculations. Commonly used algorithms employ either the diagrams' parametric representations [41] or representations as graphs. To avoid cumbersome transformations, we choose to work with the original algebraic form obtained directly from the Feynman rules.

A general L -loop scalar diagram I with P propagators has the form

$$I = \int [dl_1] \dots [dl_L] \frac{1}{D_1^{a_1} \dots D_P^{a_P}} \quad (\text{A.1})$$

with (not necessarily positive) integers a_1, \dots, a_P . The $[dl_i]$ are suitable d -dimensional integral measures, e.g. as in equation (B.1), and the propagators D_i are functions of the loop momenta

l_1, \dots, l_L , any number of external momenta, and a mass m_i . Obviously, I is invariant under a change of variables

$$\mathcal{M} : l_i \mapsto l'_i = M_{ij}l_j + q_i \quad (\text{A.2})$$

with $|\det(M)| = 1$ and constant vectors q_i .

Consider now a diagram \tilde{I} with propagators $\tilde{D}_1, \dots, \tilde{D}_P$ and the diagram I as defined by eq. (A.1). Let us denote the propagators we obtain by changing the loop momenta in I according to eq. (A.2) as D'_1, \dots, D'_P . We say that I and \tilde{I} belong to the same topology iff there is a transformation \mathcal{M} such that $\{D'_1, \dots, D'_P\} = \{\tilde{D}_1, \dots, \tilde{D}_P\}$. Likewise, I belongs to a subtopology of \tilde{I} iff for some \mathcal{M} we have $\{D'_1, \dots, D'_P\} \subseteq \{\tilde{D}_1, \dots, \tilde{D}_P\}$. The problem of mapping a diagram to a topology thus reduces to finding out whether a suitable transformation \mathcal{M} exists.

The basic idea behind our algorithm is to first look for L propagators D_i that depend on all loop momenta l_1, \dots, l_L . Then we select L appropriate mutually different target propagators \tilde{D}_{j_i} and define \mathcal{M} such that $D'_i = \tilde{D}_{j_i}$. If the sets of the remaining propagators are also equal after applying \mathcal{M} , the two topologies are the same.

To be more concrete, let us now consider a diagram I defined as in equation (A.1) with propagators of the form $D_i = p_i^2 \pm m_i^2$, where the p_i are linear combinations of loop momenta and external momenta. The generalisation to other forms of the propagators should be straightforward. In practice, we can choose the first L propagators to be of the form $D_i = l_i^2 \pm m_i^2$. The algorithm then works as follows.

1. Select a new target topology and choose a representative with propagators $\{\tilde{D}_1, \dots, \tilde{D}_P\}$ of the form $\tilde{D}_i = \tilde{p}_i^2 \pm \tilde{m}_i^2$ from it.
2. Choose a tuple $(\tilde{D}_{i_1}, \dots, \tilde{D}_{i_L})$ (that was not chosen before) of L distinct propagators with compatible masses, i.e. $\tilde{m}_{i_1} = m_1, \dots, \tilde{m}_{i_L} = m_L$. If this is not possible go back to step 1.
3. Consider the next among the 2^L transformations that map the propagators (D_1, \dots, D_L) onto $(\tilde{D}_{i_1}, \dots, \tilde{D}_{i_L})$, i.e. $l_j \mapsto \pm p_{i_j}$ $j = 1, \dots, L$. If no transformation is left go back to step 2.
4. Apply the current transformation to the propagators D_1, \dots, D_P . I then belongs to the current target topology if $\{D'_1, \dots, D'_P\} = \{\tilde{D}_1, \dots, \tilde{D}_P\}$. Else go back to step 3.

As far as identifying the topology of an integral is concerned the algorithm terminates as soon as step 4 is completed successfully. For symmetrisation we would skip step 1 and always go back from step 4 to step 3 in order to find all automorphisms.

Appendix B. Subtraction terms

Since the leading coefficients with $n = -1, 0$ in equation (6) still depend on the dimensional regulator $\epsilon = (4 - d)/2$, we first have to specify our renormalisation prescriptions in d dimensions in order to give meaningful expressions.

Our d -dimensional integration measure is given by

$$[dl_i] = \frac{d^d l_i}{i\pi^{d/2}} e^{\epsilon \gamma_E}, \quad (\text{B.1})$$

where $\gamma_E \approx 0.5772157$ is the Euler–Mascheroni constant. The counterterms in the $\overline{\text{MS}}$ scheme are now defined such that they exactly cancel the poles in ϵ . For the sake of simplicity, we refrain

from defining on-shell renormalisation and present the divergent coefficients in terms of the $\overline{\text{MS}}$ quark mass. Writing

$$\bar{c}_n^{\delta,(3)} = \sum_{i=0}^{3-n} \frac{\bar{c}_{n,i}^{\delta,(3)}}{\epsilon^i} \quad (\text{B.2})$$

we obtain for $\mu = \bar{m}$

$$\begin{aligned} \bar{c}_{-1,0}^{v,(3)} = & +1.740 C_A^2 C_F - 9.555 C_A C_F^2 + 15.433 C_F^3 \\ & - 7.803 C_A C_F T_F n_h + 7.355 C_F^2 T_F n_h \\ & - 0.228 C_A C_F T_F n_l - 1.897 C_F^2 T_F n_l \\ & - 0.935 C_F T_F^2 n_h^2 + 0.735 C_F T_F^2 n_h n_l + 1.024 C_F T_F^2 n_l^2, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \bar{c}_{-1,1}^{v,(3)} = & -1.196 C_A^2 C_F + 0.592 C_A C_F^2 - 1.377 C_F^3 \\ & + 1.130 C_A C_F T_F n_f + 0.015 C_F^2 T_F n_f + 0.009 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \bar{c}_{-1,2}^{v,(3)} = & +2.195 C_A^2 C_F + 0.649 C_A C_F^2 + 1.278 C_F^3 \\ & - 1.623 C_A C_F T_F n_f - 0.244 C_F^2 T_F n_f - 0.025 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \bar{c}_{-1,3}^{v,(3)} = & -1.058 C_A^2 C_F - 1.750 C_A C_F^2 - 0.352 C_F^3 \\ & + 0.635 C_A C_F T_F n_f + 0.531 C_F^2 T_F n_f - 0.069 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \bar{c}_{-1,4}^{v,(3)} = & +0.210 C_A^2 C_F + 0.516 C_A C_F^2 + 0.281 C_F^3 \\ & - 0.153 C_A C_F T_F n_f - 0.188 C_F^2 T_F n_f + 0.028 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \bar{c}_{0,0}^{v,(3)} = & -0.832 C_A^2 C_F - 3.606 C_A C_F^2 + 2.628 C_F^3 \\ & - 1.432 C_A C_F T_F n_h + 2.335 C_F^2 T_F n_h \\ & + 2.239 C_A C_F T_F n_l + 0.666 C_F^2 T_F n_l \\ & - 0.425 C_F T_F^2 n_h^2 - 0.479 C_F T_F^2 n_h n_l - 0.330 C_F T_F^2 n_l^2, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \bar{c}_{0,1}^{v,(3)} = & +0.277 C_A^2 C_F + 0.065 C_A C_F^2 - 0.180 C_F^3 \\ & - 0.417 C_A C_F T_F n_f + 0.172 C_F^2 T_F n_f - 0.020 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \bar{c}_{0,2}^{v,(3)} = & -0.230 C_A^2 C_F + 0.019 C_A C_F^2 \\ & + 0.150 C_A C_F T_F n_f + 0.024 C_F^2 T_F n_f - 0.017 C_F T_F^2 n_f^2, \end{aligned} \quad (\text{B.10})$$

$$\bar{c}_{0,3}^{v,(3)} = +0.070 C_A^2 C_F - 0.051 C_A C_F T_F n_f + 0.009 C_F T_F^2 n_f^2, \quad (\text{B.11})$$

$$\begin{aligned} \bar{c}_{-1,0}^{s,(3)} = & -72.707 C_A^2 C_F - 114.585 C_A C_F^2 + 20.766 C_F^3 \\ & + 14.819 C_A C_F T_F n_h + 101.776 C_F^2 T_F n_h \\ & + 62.816 C_A C_F T_F n_l + 19.095 C_F^2 T_F n_l \\ & - 17.829 C_F T_F^2 n_h^2 - 24.041 C_F T_F^2 n_h n_l - 3.175 C_F T_F^2 n_l^2, \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \bar{c}_{-1,1}^{s,(3)} = & -5.959 C_A^2 C_F + 10.188 C_A C_F^2 - 6.959 C_F^3 \\ & + 16.536 C_A C_F T_F n_h - 0.578 C_F^2 T_F n_h \end{aligned}$$

$$\begin{aligned}
& + 2.295 C_A C_F T_F n_l + 0.422 C_F^2 T_F n_l \\
& - 0.544 C_F T_F^2 n_h^2 - 0.644 C_F T_F^2 n_h n_l - 0.100 C_F T_F^2 n_l^2, \tag{B.13}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{-1,2}^{s,(3)} = & + 7.939 C_A^2 C_F + 1.310 C_A C_F^2 + 3.731 C_F^3 \\
& - 7.673 C_A C_F T_F n_h - 3.327 C_F^2 T_F n_h \\
& - 5.840 C_A C_F T_F n_l - 0.327 C_F^2 T_F n_l \\
& + 0.481 C_F T_F^2 n_h^2 + 0.296 C_F T_F^2 n_h n_l - 0.185 C_F T_F^2 n_l^2, \tag{B.14}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{-1,3}^{s,(3)} = & - 3.813 C_A^2 C_F - 11.708 C_A C_F^2 - 2.438 C_F^3 \\
& + 2.236 C_A C_F T_F n_f + 3.042 C_F^2 T_F n_f - 0.222 C_F T_F^2 n_f^2, \tag{B.15}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{-1,4}^{s,(3)} = & + 0.840 C_A^2 C_F + 4.125 C_A C_F^2 + 4.500 C_F^3 \\
& - 0.611 C_A C_F T_F n_f - 1.500 C_F^2 T_F n_f + 0.111 C_F T_F^2 n_f^2, \tag{B.16}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{0,0}^{s,(3)} = & - 1.740 C_A^2 C_F + 9.555 C_A C_F^2 - 15.433 C_F^3 \\
& + 7.803 C_A C_F T_F n_h - 7.355 C_F^2 T_F n_h \\
& + 0.228 C_A C_F T_F n_l + 1.897 C_F^2 T_F n_l \\
& + 0.935 C_F T_F^2 n_h^2 - 0.735 C_F T_F^2 n_h n_l - 1.024 C_F T_F^2 n_l^2, \tag{B.17}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{0,1}^{s,(3)} = & + 1.196 C_A^2 C_F - 0.592 C_A C_F^2 + 1.377 C_F^3 \\
& - 1.130 C_A C_F T_F n_f - 0.015 C_F^2 T_F n_f - 0.009 C_F T_F^2 n_f^2, \tag{B.18}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{0,2}^{s,(3)} = & - 2.195 C_A^2 C_F - 0.649 C_A C_F^2 - 1.278 C_F^3 \\
& + 1.623 C_A C_F T_F n_f + 0.244 C_F^2 T_F n_f + 0.025 C_F T_F^2 n_f^2, \tag{B.19}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{0,3}^{s,(3)} = & + 1.058 C_A^2 C_F + 1.750 C_A C_F^2 + 0.352 C_F^3 \\
& - 0.635 C_A C_F T_F n_f - 0.531 C_F^2 T_F n_f + 0.069 C_F T_F^2 n_f^2, \tag{B.20}
\end{aligned}$$

with $n_f = n_h + n_l$.

In addition to the listed coefficient $\bar{C}_{-1}^{v,(3)}$ we require the corresponding coefficient $\bar{C}_{-1,\text{diag}}^{a,(3)}$ in the low-energy expansion of the flavour diagonal axial-vector correlator in order to compute the correction to the ρ parameter. Since the pole parts of these two coefficients have to cancel, we can decompose the latter coefficient as

$$\bar{C}_{-1,\text{diag}}^{a,(3)} = \bar{C}_{-1,\text{diag}}^{a,(3)} \Big|_{\text{fin}} - \sum_{i=1}^4 \frac{\bar{c}_{-1,i}^{v,(3)}}{\epsilon^i} \tag{B.21}$$

with the coefficients $\bar{c}_{-1,i}^{v,(3)}$ as in equations (B.4)–(B.7). The remaining finite part is given by

$$\begin{aligned}
\bar{C}_{-1,\text{diag}}^{a,(3)} \Big|_{\text{fin}} = & + 2.484 C_A^2 C_F - 8.319 C_A C_F^2 + 16.954 C_F^3 \\
& - 5.300 C_A C_F T_F n_h + 2.759 C_F^2 T_F n_h \\
& - 1.598 C_A C_F T_F n_l - 4.210 C_F^2 T_F n_l \\
& - 0.247 C_F T_F^2 n_h^2 + 1.585 C_F T_F^2 n_h n_l + 1.492 C_F T_F^2 n_l^2. \tag{B.22}
\end{aligned}$$

Appendix C. Supplementary material

Supplementary material related to this article can be found online at <http://dx.doi.org/10.1016/j.nuclphysb.2015.07.031>.

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